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An alternative derivation of the Lense-Thirring drag on the orbit of a test body

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Summary. — In the weak field and slow motion approximation of general relativity a new approach in deriving the secular Lense-Thirring effect on the orbital elements of a test body in the external field of different central rotating sources exhibiting axial symmetry is presented. The approach adopted in the present work, in the case of a perfectly spherical source, leads to the well known Lense-Thirring formulas for all the Keplerian orbital elements of the freely falling particle. The corrections induced in the case of a central nonspherical, axisymmetric body are also worked out.

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1. – Introduction.

Latest years have seen increasing efforts devoted to the measurement of the general relativistic Lense-Thirring effect [1, 2] in the weak gravitational field of the Earth by means of artificial satellites. At present, there are two main proposals which point towards the implementation of this goal: the Gravity Probe-B mission [3], and the approach proposed in [4] which consists in using the actually orbiting LAGEOS laser-ranged satellite and launching another satellite of LAGEOS type, the LARES, with the same orbital parameters of LAGEOS except for the inclination which should be supplementary with respect to it. At present, both these two satellites have not yet been launched: however, while the GP-B is scheduled to fly in 2002, to date the LARES mission has not been approved by any Spatial Agency.

Recently Ciufolini [5] has put forward an alternative strategy based on the utilization of the already existing LAGEOS and LAGEOS II which could allow the detection of the Lense-Thirring drag at a precision level of 20 % [6, 7, 8]. While the GP-B mission is focused on the gravitomagnetic dragging of the spin of a freely falling body, in the LAGEOS experiment it is the whole orbit of the satellite which is considered to undergo the secular Lense-Thirring precession. More exactly, among the Keplerian orbital elements

the node and the perigee are affected by the gravitomagnetic force. For LAGEOS and LAGEOS II its effect consists in secular precessions amounting to:

$$\dot{\Omega}_{\rm LT}^{\rm LAGEOS} \simeq 31 \text{ mas/y}, \tag{1}$$
$$\dot{\Omega}_{\rm LT}^{\rm LAGEOSII} \simeq 31.5 \text{ mas/y}, \tag{2}$$
$$\dot{\omega}_{\rm LT}^{\rm LAGEOSII} \simeq -57 \text{ mas/y}, \tag{3}$$

where mas/y stands for milliarcseconds per year([1])

In this work it is presented an alternative strategy in order to derive the secular gravitomagnetic precessions which reveals itself useful especially in the prediction of the behavior of all the Keplerian orbital elements of the test body in the gravitational field of different kinds of central rotating sources. Indeed, the calculations involve not only the secular effects for a perfectly spherical source, which lead to eqs.(1)-(3), but also for a central body which is only axially symmetric.

Until now the general relativistic motion in the field of a non spherical central body has been treated in [9, 10] for a static, nonrotating source and in [11, 12, 13] for a rotating source in the context of the GP-B mission. The influence of the non sphericity of the central source on the gravitomagnetic clock effect [14] is investigated in [15].

Our calculations are based on the use of the Lagrangian planetary equations [16] and a noncentral hamiltonian term whose existence, if from one hand can be rigorously deduced, from the other hand can be intuitively guessed by analogy from the corresponding term in electrodynamics for a charged particle acted upon by electric and magnetic fields.

2. – The gravitomagnetic potential

In general, it can be proved that the general relativistic equations of motion of a test particle of mass m freely falling in a stationary gravitational field, in the weak field and slow motion approximation, are given by [15]:

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \simeq m(\mathbf{E}_g + \frac{1}{c}\mathbf{v}\times\mathbf{B}_g). \tag{4}$$

In eq.(4) c is the speed of light *in vacuo* while $\mathbf{E}_g$ and $\mathbf{B}_g$ are the gravitoelectric and the gravitomagnetic fields, respectively.

If a perfectly spherically symmetric rotating body is assumed as gravitational source, in eq.(4) $\mathbf{E}_g = -GM\mathbf{r}/r^3$ is the Newtonian gravitational field of a spherical body, with M its mass and G the Newtonian gravitational constant, while $\mathbf{B}_g$ is given by:

$$\mathbf{B}_g = \nabla\times\mathbf{A}_g \simeq 2\frac{G}{c}\,[\frac{\mathbf{J}-3(\mathbf{J}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}}{r^3}], \tag{5}$$

in which:

$$\mathbf{A}_g(\mathbf{r}) \simeq -2\frac{G}{c}\frac{\mathbf{J}\times\mathbf{r}}{r^3}. \tag{6}$$

([1]) The perigee of the LAGEOS has not been considered since its rate is hard to measure [5].

In eqs.(5)-(6) $\mathbf{J}$ is the angular momentum of the central body. The field $\mathbf{A}_g$, named gravitomagnetic potential, is due to the off-diagonal components of the spacetime metric tensor:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ \ \mu, \nu = 0, 1, 2, 3, \tag{7}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor and $(\mathbf{A}_g)_k \equiv h_{0k},\ \ k = 1, 2, 3$. In obtaining eq.(6) a non rotating reference frame $K\{x,\ y,\ z\}$ with the z axis directed along $\mathbf{J}$ and the $\{x,\ y\}$ plane coinciding with the equatorial plane of the gravitational source has been used. The origin is located at the center of mass of the central body.

The gravitomagnetic potential generates a non central gravitational contribute due uniquely to the angular momentum of the gravitational source that the Newtonian mechanics does not predict, though the conditions of validity of eq.(4) are the same for which the latter holds as well. So it is possible to speak of mass-energy currents whose motion exerts a non central gravitational force on a test massive body analogous to the Lorentz force felt by a charged particle of charge q and mass m when it is moving in a electromagnetic field. Indeed, its equations of motions:

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \simeq q(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}) \tag{8}$$

are formally analogous to eq.(4). Eq.(8) can be derived by means of the Lagrangian:

$$\mathcal{L}_{e.m.} = \frac{1}{2}mv^2 - qV + \frac{q}{c}(\mathbf{v}\cdot\mathbf{A}), \tag{9}$$

where $\mathbf{v}$ is the velocity of the particle while V and $\mathbf{A}$ are the scalar and vector potential, respectively, of the electromagnetic field.

Since one of the most promising way to detect the gravitomagnetic precession consists in employing artificial Earth satellites, it would be helpful to derive the rate equations for the change in the parameters that characterize their orbits. To this aim one could introduce “by hand” a perturbative term $k\ (\mathbf{v}\cdot\mathbf{A}_g)$ in the gravitational Lagrangian of the particle and use it in some particular perturbative scheme; the constant k would be determined by means of dimensional considerations and taking in account that it should be built up of universal constants. In fact it is possible to show that a non central term analogous to $\frac{q}{c}(\mathbf{v}\cdot\mathbf{A})$ can be rigorously deduced in the Lagrangian of a test body in the gravitational field of a spinning mass-energy distribution, and that it can be exploited in deriving straightforwardly the effect of the gravitomagnetic potential on the Keplerian orbital elements of the test body.

3. – The rate equations for the Keplerian orbital elements

The relativistic Lagrangian for a free particle of mass m in a gravitational field can be cast into the form:

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)}. \tag{10}$$

In eq.(10) the term $\mathcal{L}^{(1)}$ refers to the off-diagonal terms of the metric:

$$\mathcal{L}^{(1)} = \frac{m}{c}g_{0k}\dot{x^0}\dot{x^k}. \tag{11}$$

In this case, recalling that the slow motion approximation is used, eq.(11) becomes [15]:

$$\mathcal{L}^{(1)} \equiv \mathcal{L}_{gm} \simeq \frac{m}{c}\ (\mathbf{A}_g \cdot \mathbf{v}). \tag{12}$$

In this paper it is proposed to adopt $\mathcal{L}_{gm}$ given by eq.(12), with $\mathbf{A}_g$ given by eq.(6), in order to derive the Lense-Thirring effect on the orbital elements of a particle in the field of a rotating gravitational source.

To this aim it must be assumed that under the gravitomagnetic force the departures of the test body' s trajectory from the unperturbed Keplerian ellipse are very small in time. This allows to introduce the concept of osculating ellipse. So the perturbed motion can be described in terms of unperturbed Keplerian elements varying in time. Consider the frame $K\{x,\ y,\ z\}$ previously defined and a frame $K^{'}\{X,\ Y,\ Z\}$ with the Z axis directed along the orbital angular momentum $\mathbf{l}$ of the test body, the plane $\{X,\ Y\}$ coinciding with the orbital plane of the test particle and the X axis directed toward the pericenter. $K\{x,\ y,\ z\}$ and $K^{'}\{X,\ Y,\ Z\}$ have the same origin located in the center of mass of the central body. The Keplerian orbital elements are:

• $a,\ e$

The semimajor axis and the ellipticity which define the size and the shape of the orbit in its plane.

• $\Omega,\ i$

The longitude of the ascending node and the inclination which fix the orientation of the orbit in the space, i.e. of $K^{'}\{X,\ Y,\ Z\}$ with respect to $K\{x,\ y,\ z\}$. The longitude of the ascending node Ω is the angle in the equatorial plane of $K\{x,\ y,\ z\}$ between the x axis and the line of nodes in which the orbital plane intersects the equatorial plane. The inclination i is the angle between z and Z axis.

• $\omega,\ \mathcal{M}$

The argument of pericenter and the mean anomaly. The argument of pericenter ω is the angle in the orbital plane between the line of nodes and the X axis; it defines the orientation of the orbit in its plane. The mean anomaly $\mathcal{M}$ specifies the motion of the test particle on its orbit. It is related to the mean motion $n = (GM)^{1/2}a^{-3/2}$ through $\mathcal{M} = n(t - t_p)$ in which t_p is the time of pericenter passage.

It is customary to define also the longitude of pericenter

• $\varpi = \Omega + \omega$,

the argument of latitude

• $u = \omega + f$

where f is the angle, called true anomaly, which in the orbital plane determines the position of the test particle with respect to the pericenter, and the mean longitude at the epoch t_0

• $\varepsilon = \varpi + n(t_0 - t_p)$. If $t_0 = 0$, it is customary to write ε as $L_0 = \varpi - nt_p$.

The matrix $\mathbf{R}_{xX}$ for the change of coordinates from $K^{'}\{X,\ Y,\ Z\}$ to $K\{x,\ y,\ z\}$ is:

$$\begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{pmatrix}. \tag{13}$$

Using eq.(13) and $X = r\cos f,\ \ Y = r\sin f,\ \ Z = 0$ it is possible to express the geocentric

rectangular Cartesian coordinates of the orbiter in terms of its Keplerian elements:

$$\left\{\begin{array}{l} x = r(\cos u \cos\Omega - \sin u \cos i \sin\Omega) \\ y = r(\cos u \sin\Omega + \sin u \cos i \cos\Omega) \\ z = r \sin u \sin i. \end{array}\right. \tag{14}$$

Redefining suitably the origin of the angle Ω so that $\cos\Omega = 1$, $\sin\Omega = 0$, the previous equations become:

$$\left\{\begin{array}{l} x = r\cos u \\ y = r\sin u \cos i \\ z = r \sin u \sin i. \end{array}\right. \tag{15}$$

Considering for the test particle the total mechanical energy with the sign reversed, according to [16], $\mathcal{F} \equiv -\mathcal{E}_{tot} = -(\mathcal{T} + \mathcal{U})$, where $\mathcal{T}$ and $\mathcal{U}$ are the kinetic and potential energies per unit mass, it is possible to work out the analytical expressions for the rate of changes of $a,\ e,\ i,\ \Omega,\ \omega,\ \mathcal{M}$ due to any non central gravitational contribution. To this aim it is useful isolating in $\mathcal{U}$ the central part $-\mathcal{C}$ of the gravitational field from the terms $-\mathcal{R}$ which may cause the Keplerian orbital elements to change in time: $\mathcal{U} = -\mathcal{C} - \mathcal{R}$. In this way $\mathcal{F}$ becomes:

$$\mathcal{F} = \frac{GM}{r} + \mathcal{R} - \mathcal{T} = \frac{GM}{2a} + \mathcal{R}. \tag{16}$$

Concerning the perturbative scheme to be employed, the well known Lagrange planetary equations are adopted. At first order, they are:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial\mathcal{R}}{\partial\mathcal{M}}, \tag{17}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1-e^2}{na^2e}\frac{\partial\mathcal{R}}{\partial\mathcal{M}} - \frac{(1-e^2)^{1/2}}{na^2e}\frac{\partial\mathcal{R}}{\partial\omega}, \tag{18}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \cos i\frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial\mathcal{R}}{\partial\omega} - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial\mathcal{R}}{\partial\Omega}, \tag{19}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial\mathcal{R}}{\partial i}, \tag{20}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\cos i\frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial\mathcal{R}}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2e}\frac{\partial\mathcal{R}}{\partial e}, \tag{21}$$

$$\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t} = n - \frac{1-e^2}{na^2e}\frac{\partial\mathcal{R}}{\partial e} - \frac{2}{na}\frac{\partial\mathcal{R}}{\partial a}. \tag{22}$$

The idea of this work consists in using $\mathcal{L}_{gm}$ to obtain a suitable non central term $\mathcal{R}_{gm}$ to be employed in these equations. This can be done considering the Hamiltonian for the test particle:

$$\mathcal{H} = \mathbf{p}\cdot\mathbf{v} - \mathcal{L}. \tag{23}$$

Inserting eq.(10) in eq.(23) one has:

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}_{gm}, \tag{24}$$

with $\mathcal{H}_{gm} = -\frac{m}{c}\ (\mathbf{A}_g \cdot \mathbf{v})$. So it can be posed:

$$\mathcal{R}_{gm} = -\frac{\mathcal{H}_{gm}}{m} = \frac{1}{c}\ (\mathbf{A}_g \cdot \mathbf{v}). \tag{25}$$

Now it is useful to express eq.(25) in terms of the Keplerian elements. Referring to eq.(6), eq.(15), recalling that in the frame $K\{x,\ y,\ z\}$ $\mathbf{J} = (0,\ 0,\ J)$ and that for an unperturbed Keplerian motion:

$$\frac{1}{r} = \frac{(1 + e\cos f)}{a(1-e^2)}, \tag{26}$$

it is possible to obtain, for a perfectly spherical central body:

$$\mathcal{R}_{gm} = -\frac{2G}{c^2}\frac{J\cos i}{r}\dot{u} = -\frac{2GJ\cos i}{c^2}\frac{(1+e\cos f)}{a(1-e^2)}\dot{u}. \tag{27}$$

In eq.(27) $\dot{u} \simeq \dot{f}$ is assumed due to the the fact that the osculating element ω may be retained almost constant on the temporal scale of variation of the true anomaly of the test body.

4. – Secular gravitomagnetic effects on the Keplerian orbital elements: spherical central source

The secular effects can be worked out by adopting the same strategy followed in [16] for a similar kind of perturbing functions. When eq.(27) is averaged over one orbital period P of the test body, $a,\ e,\ i,\ \Omega$ and ω are to be considered constant:

$$<\mathcal{R}>_{2\pi} = -\frac{1}{P}\int_0^P \frac{G}{c^2}\frac{2J}{r}\cos i du = -2n\frac{G}{c^2}\frac{J\cos i}{2\pi}\int_0^{2\pi}\frac{df}{r} = -2n\frac{G}{c^2}\frac{J\cos i}{a(1-e^2)}. \tag{28}$$

The relation $du = d\omega + df = df$ has been used in eq.(28) which can now be used in determining the secular changes of the Keplerian orbital elements of the test body. It yields:

$$\frac{\partial <\mathcal{R}>_{2\pi}}{\partial \mathcal{M}} = 0, \tag{29}$$

$$\frac{\partial <\mathcal{R}>_{2\pi}}{\partial \omega} = 0, \tag{30}$$

$$\frac{\partial <\mathcal{R}>_{2\pi}}{\partial \Omega} = 0, \tag{31}$$

$$\frac{\partial <\mathcal{R}>_{2\pi}}{\partial i} = 2n\frac{G}{c^2}\frac{J\sin i}{a(1-e^2)}, \tag{32}$$

$$\frac{\partial <\mathcal{R}>_{2\pi}}{\partial e} = -4n\frac{G}{c^2}\frac{Je\cos i}{a(1-e^2)^2}. \tag{33}$$

A particular care is needed for the treatment of n when the derivative of $<\mathcal{R}>_{2\pi}$ with respect to a is taken; indeed, it must be posed as:

$$\frac{\partial<\mathcal{R}>_{2\pi}}{\partial a}=\frac{\partial<\mathcal{R}>_{2\pi}}{\partial a}_{|n}+\frac{\partial<\mathcal{R}>_{2\pi}}{\partial n}_{|a}\frac{\partial n}{\partial a}. \tag{34}$$

In eq.(34)

$$\frac{\partial<\mathcal{R}>_{2\pi}}{\partial a}_{|n}=2n\frac{G}{c^2}\frac{J\cos i}{a^2(1-e^2)}, \tag{35}$$

and

$$\frac{\partial<\mathcal{R}>_{2\pi}}{\partial n}_{|a}\frac{\partial n}{\partial a}=3n\frac{G}{c^2}\frac{J\cos i}{a^2(1-e^2)}. \tag{36}$$

From eq.(17) and eq.(29) it appears that there are no secular changes in the semimajor axis, and so the orbital period of the test body, related to the mean motion by $P=2\pi/n$, can be considered constant. This implies that in eq.(34) only eq.(35) must be retained. Using eqs.(29)-(34) in eqs.(17)-(22) one obtains for the secular rates:

$$\left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{\mathrm{LT}}=0, \tag{37}$$

$$\left.\frac{\mathrm{d}e}{\mathrm{d}t}\right|_{\mathrm{LT}}=0, \tag{38}$$

$$\left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{\mathrm{LT}}=0, \tag{39}$$

$$\left.\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right|_{\mathrm{LT}}=\frac{G}{c^2}\frac{2J}{a^3(1-e^2)^{3/2}}, \tag{40}$$

$$\left.\frac{\mathrm{d}\omega}{\mathrm{d}t}\right|_{\mathrm{LT}}=-\frac{G}{c^2}\frac{6J}{a^3(1-e^2)^{3/2}}\cos i, \tag{41}$$

$$\left.\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t}\right|_{\mathrm{LT}}=0. \tag{42}$$

They are the well known Lense-Thirring equations [1, 2]. In deriving them it has been assumed that the spatial average over f yields the same results for the temporal average [17].

5. – Secular gravitomagnetic effects on the Keplerian orbital elements: non-spherical central source

In this Section we shall deal with a non spherical central rotating source with axial symmetry around the rotation axis.

In classical electrodynamics the potential vector for a generic steady current distribution can be written as:

$$\mathbf{A}(\mathbf{r})=\frac{1}{c}\int\frac{\rho(\mathbf{r}^{'})\mathbf{v}}{|\ \mathbf{r}-\mathbf{r}^{'}|}d\mathbf{r}^{'}. \tag{43}$$

The quantity ρ is the charge density which, in general, depends on $\mathbf{r}$, but, in this case, not on time. If we assume that the current distribution rotates uniformly around an axis, chosen as z axis, then, since for any current element $\mathbf{v} = \alpha\hat{z} \times \mathbf{r}$ with α angular velocity of the current distribution eq.(43) becomes:

$$\mathbf{A}(\mathbf{r}) = \frac{\alpha}{c}\int \frac{\rho(\mathbf{r}^{'})\hat{z} \times \mathbf{r}}{\mid \mathbf{r} - \mathbf{r}'\mid} d\mathbf{r}^{'}. \tag{44}$$

In order to deal with an axisymmetric current distribution let us introduce the cylindrical coordinates:

$$\left\{ \begin{array}{l} x = \xi\cos\phi \\ y = \xi\sin\phi \\ z = z, \end{array} \right. \tag{45}$$

$\xi \geq 0,\ 0 \leq \phi < 2\pi$. In this case it can be shown that the lines of the potential vector are circles around the z axis and its modulus is a cylindrical symmetric function of ξ and z. So, eq.(44) reduces to:

$$\mathbf{A}(\mathbf{r}) = \frac{\alpha}{c}H(\xi,\ z)\mathbf{e}_{\phi}. \tag{46}$$

The function $H(\xi,\ z)$ remains unchanged under rotations around the z axis. Passing from cylindrical to rectangular Cartesian coordinates, the components of the potential vector become:

$$A_1 = -\frac{\alpha}{c}H(\xi,\ z)y, \tag{47}$$

$$A_2 = \frac{\alpha}{c}H(\xi,\ z)x, \tag{48}$$

$$A_3 = 0. \tag{49}$$

Recalling that, in the linear approximation, the general relativistic gravitomagnetic potential can be obtained from the vector potential of electromagnetism times $-4G$ [2], eqs.(47)-(49) lead to:

$$h_{01} = \frac{4G\alpha}{c}H(\xi,\ z)y, \tag{50}$$

$$h_{02} = -\frac{4G\alpha}{c}H(\xi,\ z)x, \tag{51}$$

$$h_{03} = 0. \tag{52}$$

Of course, eqs.(50)-(52) can be rigorously obtained solving the linearized Einstein field equation, written in the Lorenz gauge, for a localized ($g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ at spatial infinity), axisymmetric, stationary mass distribution in the weak field and slow motion approximation:

$$\Delta h_{0k} = 0,\ k = 1, 2, 3 \tag{53}$$

$$\Delta h_{0k} = \frac{16\pi G}{c^4}T_{0k},\ k = 1, 2, 3. \tag{54}$$

The quantities T_{0k} are the $\{0k\}$ components of the stress-energy tensor for the matter; in deriving eq.(54) the internal stresses have been neglected. Eq.(54) is valid inside the matter, while eq.(53) holds in the free space outside the central body and tells us that $H(\xi,\ z)y$ and $H(\xi,\ z)x$ are harmonic functions. This feature was used by Teyssandier in obtaining a multipolar expansion of $H(\xi,\ z)$ [11]. Introducing in the frame K the usual spherical coordinates $\{r,\ \theta,\ \phi\}$, if r_e is the radius of the smallest sphere centered on the origin of the coordinates containing the whole body (in practice, it should be the equatorial radius, $R_{\oplus}$ in the case of the Earth), in the region $r \geq r_1 > r_e$ $H(r,\ \theta)$ is given by:

$$H(r,\ \theta) = \frac{I}{2r^3}\ [1 - \sum_{l=1}^{\infty} K_l(\frac{r_e}{r})^l P^{'}_{l+1}(\cos\theta)], \tag{55}$$

with:

$$K_l = \frac{2}{2l+3}\ \frac{Mr_e^2}{I}(L_l - J_{l+2}), \tag{56}$$

$$L_l = -\frac{1}{Mr_e^{l+2}}\int \rho(r^{'},\ \theta^{'})r^{'\,l+2} P_l(\cos\theta^{'})d\mathbf{r}^{'}. \tag{57}$$

In eqs.(55)-(57) I is the moment of inertia of the body about the z axis, M is the total mass of the central body, ρ is its density, $P^{'}_{l+1}(\cos\theta)$ is the first derivative of the Legendre polynomial of degree $l+1$ and J_l is the Newtonian multipole moment of degree l, given by:

$$J_l = -\frac{1}{Mr_e^l}\int \rho(r^{'},\ \theta^{'})r^{'\,l} P_l(\cos\theta^{'})d\mathbf{r}^{'}. \tag{58}$$

It is interesting to note that, if in eq.(55) only the spherical symmetric term:

$$H^{(0)}(r) = \frac{I}{2r^3} \tag{59}$$

is retained, as it would be the case for a perfectly spherical central body, eqs.(50)-(52) reduce to:

$$h^{(0)}_{01} = \frac{2GI\alpha}{c}\frac{y}{r^3}, \tag{60}$$

$$h^{(0)}_{02} = -\frac{2GI\alpha}{c}\frac{x}{r^3}, \tag{61}$$

$$h^{(0)}_{03} = 0. \tag{62}$$

For a spherical rotating body $J = I\alpha$, and so eqs.(60)-(62) can be cast into the familiar form:

$$\mathbf{A}^{(0)}_g = -\frac{2G}{c}\frac{\mathbf{J}\times\mathbf{r}}{r^3}. \tag{63}$$

The correction of order l to the gravitomagnetic potential due to the nonsphericity of the central rotating body is given by:

$$h_{01}^{(l)} = -\frac{2GI\alpha}{r^3c^3}\ y\ K_l(\frac{r_e}{r})^l P_{l+1}^{'}(\cos\theta), \tag{64}$$

$$h_{02}^{(l)} = \frac{2GI\alpha}{r^3c^3}\ x\ K_l(\frac{r_e}{r})^l P_{l+1}^{'}(\cos\theta), \tag{65}$$

$$h_{03}^{(l)} = 0. \tag{66}$$

Modeling the central body as a spheroid stratified into ellipsoidal shells, in [11] is shown that all the relativistic coefficients K_l are null, except K_2; if we consider our planet, from the analysis of Teyssandier of various models of Earth's interior, it results to be positive and of $\mathcal{O}(10^{-3})$.

The starting point in deriving the relativistic multipole corrections of degree l to eqs.(37)-(42) is the perturbative lagrangian term:

$$\mathcal{L}_{gm}^{(l)} = \frac{m}{c}(\mathbf{A}_g^{(l)}\cdot\mathbf{v}), \tag{67}$$

where m is the mass of the point particle and $\mathbf{v}$ is its velocity. In the case $l = 2$, in eqs.(64)-(66) the first derivative of the Legendre polynomial of degree 3 appear;

$$P_3(q) = \frac{1}{2}\ (5q^3 - 3q),\ \ q = \cos\theta, \tag{68}$$

$$P_3^{'}(q) = \frac{1}{2}\ (15q^2 - 3),\ \ q = \cos\theta. \tag{69}$$

Inserting eq.(69) in eqs.(64)-(66) for $l = 2$ leads to:

$$\mathcal{L}_{gm}^{(2)} = -m\frac{GI\alpha}{c^2r^3}\ \frac{(y\dot{x} - x\dot{y})}{r^2}\ r_e^2K_2(15\cos^2\theta - 3). \tag{70}$$

Such a perturbative term must be expressed in term of the Keplerian orbital elements in order to be employed in the first order Lagrange planetary equations. In this case $\mathcal{R}$ is $\mathcal{R}^{(2)} = \frac{1}{c}(\mathbf{A}_g^{(2)}\cdot\mathbf{v})$. Using eq.(15) and recalling that $z = r\cos\theta$ it is possible to obtain:

$$\frac{(y\dot{x} - x\dot{y})}{r^2} = -\dot{u}\cos i \tag{71}$$

$$\cos\theta = \sin u \sin i. \tag{72}$$

Neglecting terms of order $\mathcal{O}(e^n),\ n \geq 2$, we can write:

$$\frac{1}{r^3} \simeq \frac{1 + 3e\cos f}{a^3(1 - e^2)^3}, \tag{73}$$

having assumed:

$$\frac{1}{r} = \frac{1 + e\cos f}{a(1 - e^2)}. \tag{74}$$

Eqs.(71)-(73) in eq.(70) give:

$$\mathcal{R}^{(2)} = \frac{GI\alpha}{c^2}\frac{r_e^2 K_2 \cos i}{a^3(1-e^2)^3}(1+3e\cos f)(15\sin^2 u \sin^2 i - 3)\dot{u}. \tag{75}$$

If we want to investigate the secular trends of the orbital elements, we must average eq.(75) over an orbital period of the test body, as done in the previous Section. It is straightforward to obtain:

$$< (1+3e\cos f)(15\sin^2 u \sin^2 i - 3)\ \dot{u} >_{2\pi} = \frac{n}{2}(9 - 15\cos^2 i). \tag{76}$$

In deriving eq.(76) we have adopted the reasonable assumption that the pericenter of the test body remains almost unchanged during an orbital revolution, i. e. $du = d\omega + df \simeq df$. So we have:

$$< \mathcal{R}^{(2)} >_{2\pi} = n\frac{GI\alpha}{2c^2}\frac{r_e^2 K_2}{a^3(1-e^2)^3}(9\cos i - 15\cos^3 i). \tag{77}$$

Eq.(77) can be considered as the 2nd order correction to the gravitomagnetic perturbing function given in eq.(28). Eq.(77) yields:

$$\frac{\partial < \mathcal{R}^{(2)} >_{2\pi}}{\partial \mathcal{M}} = 0, \tag{78}$$

$$\frac{\partial < \mathcal{R}^{(2)} >_{2\pi}}{\partial \omega} = 0, \tag{79}$$

$$\frac{\partial < \mathcal{R}^{(2)} >_{2\pi}}{\partial \Omega} = 0, \tag{80}$$

$$\frac{\partial < \mathcal{R}^{(2)} >_{2\pi}}{\partial i} = n\frac{GI\alpha}{2c^2}\frac{r_e^2 K_2}{a^3(1-e^2)^3}\sin i(45\cos^2 i - 9), \tag{81}$$

$$\frac{\partial < \mathcal{R}^{(2)} >_{2\pi}}{\partial e} = n\frac{3eGI\alpha}{c^2}\frac{r_e^2 K_2}{a^3(1-e^2)^4}(9\cos i - 15\cos^3 i), \tag{82}$$

$$\frac{\partial < \mathcal{R}^{(2)} >_{2\pi}}{\partial a}_{|n} = -n\frac{3GI\alpha}{2c^2}\frac{r_e^2 K_2}{a^4(1-e^2)^3}(9\cos i - 15\cos^3 i). \tag{83}$$

By inserting eqs.(78)-(83) in eqs.(17)-(22) it can be obtained for the secular rates:

$$\left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{\mathrm{LT}}^{(2)} = 0, \tag{84}$$

$$\left.\frac{\mathrm{d}e}{\mathrm{d}t}\right|_{\mathrm{LT}}^{(2)} = 0, \tag{85}$$

$$\left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{\mathrm{LT}}^{(2)} = 0, \tag{86}$$

$$\left.\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right|_{\mathrm{LT}}^{(2)} = \frac{GI\alpha}{2c^2}\frac{r_e^2 K_2}{a^5(1-e^2)^{7/2}}(45\cos^2 i - 9), \tag{87}$$

$$\left.\frac{\mathrm{d}\omega}{\mathrm{d}t}\right|_{\mathrm{LT}}^{(2)} = -\cos i \left.\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right|_{\mathrm{LT}}^{(2)} + \frac{3GI\alpha}{c^2}\frac{r_e^2 K_2}{a^5(1-e^2)^{7/2}}(9\cos i - 15\cos^3 i), \tag{88}$$

$$\left.\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t}\right|_{\mathrm{LT}}^{(2)} = 0. \tag{89}$$

Eqs.(84)-(89) yield the corrections to the precessional rate Lense-Thirring equations when the central rotating body is not perfectly spherical but only axially symmetric.

If we use in eqs.(87)-(88) the values $r_e = R_\oplus \simeq 6378$ km, $K_2 = 0.874 \times 10^{-3}$ [12] we can obtain an estimate of the sensitivity of LAGEOS and LAGEOS II to the relativistic Earth's quadrupole correction to the Lense-Thirring precessional rates. The results are:

$$\dot{\Omega}^{(2)}_{\mathrm{LAGEOS}} = -6.7 \times 10^{-1} \text{ mas/century}, \tag{90}$$

$$\dot{\Omega}^{(2)}_{\mathrm{LAGEOSII}} = 2.8 \text{ mas/century}, \tag{91}$$

$$\dot{\omega}^{(2)}_{\mathrm{LAGEOSII}} = 5.4 \text{ mas/century}. \tag{92}$$

The present accuracy in the measurement of the LAGEOSs' rates of the node and the perigee is of the order of 1 mas/y. This implies that such tiny corrections do not affect the current efforts in detecting the Lense-Thirring drag.

6. – Conclusions

In the weak field and slow motion approximation of general relativity we have developed an alternative approach to the calculation of the Lense-Thirring effect on the Keplerian orbital elements of a test particle freely falling in the field of different kinds of axially symmetric central sources.

Such strategy stresses the formal analogy with the electromagnetism recalling the lagrangian of a charged particle acted upon by the Lorentz force and create a link to space geodesy by exploiting the widely used Lagrangian planetary equations.

For a perfectly spherical source the well known Lense-Thirring precessional rates for the node and the perigee are obtained.

If departures from sphericity of the gravitational source are accounted for, it has been found that only the node and the perigee are affected by them through additional secular rates.

At present they cannot influence the current measurement of the Lense-Thirring drag by means of LAGEOS and LAGEOS II because they fall below the experimental sensitivity.

* * *

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